The random phase approximation is an approximation made in the study of interacting fermions or bosons. It could be thought of as the linear-response approximation for the response of the system to an external potential, \( \mathcal{H} \supset V(r)\psi^\dagger(r)\psi(r) \), which is how we learned it in Physics 295b. It can also be thought of as a preservation of all quadratic terms in the action, which is how we learned it from Kamenev Field Theory of Nonequilibrium Systems.

In the seminal paper Phys. Rev. 92 609 (1953), Bohm and Pines introduced a random phase, which is historically why we call this idea “RPA.” Here, we follow Stefanucci and van Leeuwen, Nonequilibrium Many-Body Theory of Quantum Systems, and do it a different way which still tells us why we call it RPA. It is based on the density response function, \( \chi \). We will see that RPA has to do with the interaction part of \( \mathcal{H} \), and that actually random phase approximation is intimately related to stationary phase approximation (or saddle-point approximation).

1 Problem set-up

Consider the interacting energy for fermions,

\[
\mathcal{H} = -\frac{1}{2} \int_{x\sigma} \psi_\sigma^\dagger(x) \nabla^2 \psi_\sigma(x) + \frac{1}{2} \int_{xx'\sigma\sigma'} v(x - x') \psi_\sigma^\dagger(x) \psi_\sigma^\dagger(x') \psi_\sigma(x) \psi_\sigma(x').
\]

The transform conventions

\[
\psi_\sigma(x) = \frac{1}{V} \sum_p e^{ip \cdot x} c_{p\sigma}, \quad \psi_\sigma^\dagger(x) = \frac{1}{V} \sum_p e^{-i p \cdot x} c_{p\sigma}^\dagger
\]

bring this to the form

\[
\mathcal{H} = \frac{1}{V} \sum_{p\sigma} (\epsilon_p - \frac{v(0)}{2}) c_{p\sigma}^\dagger c_{p\sigma} + \frac{1}{2V} \sum_q v(q) \rho_q \rho_{-q}.
\]

The density response function is defined in momentum space to be

\[
\chi(r, t; r', t') = \frac{1}{V^2} \sum_{pq} e^{iq \cdot (r-r')} \chi_p(q, t - t'), \quad \text{where}
\]
\[ \chi_p(q, t - t') = \frac{1}{V} \sum_\sigma -i\theta(t - t') \langle [\rho_{pq\sigma}(t), \rho_{-q}(t')] \rangle. \]

This is useful because an external potential couples to the density, like \( H_{\text{ext}} \sim V(r)\rho(r, t) \). If we compute this commutator, the \( V(r) \) just comes along for the ride, and then we have effectively already computed \([H_{\text{ext}}, \rho]\), which is necessary for the equations of motion method, for example.

### 2 Equation of motion for \( \chi \)

Let us study the equation of motion for \( \chi_p(q, t) \). We need to differentiate with respect to \( t \), on both the Heaviside function and the operators. When we apply \( \frac{d}{dt} \) to the operators, we actually turn it into the equation of motion \( \frac{dO}{dt} = i[H, O] \) in Heisenberg picture.

The result is
\[
 i\frac{d\chi_p(q, t)}{dt} = \frac{1}{V} \sum_\sigma \delta(t) \langle [\rho_{pq\sigma}(t), \rho_{-q}(0)] \rangle + \frac{1}{V} \sum_\sigma -i\theta(t) \langle [\rho_{pq\sigma}(t), H], \rho_{-q}(0) \rangle.
\]

The first term is easy, \([\rho_{pq\sigma}, \rho_{-q}] = \sum_\sigma (c_{p\sigma}^\dagger c_{p\sigma} - c_{p+q\sigma}^\dagger c_{p+q\sigma})\).

The second term has two parts. The commutator with the kinetic term is
\[
[\rho_{pq\sigma}, \frac{1}{V} \sum_\sigma (\epsilon_p - \frac{v(0)}{2}) c_{p\sigma}^\dagger c_{p\sigma}] = (\epsilon_{p+q} - \epsilon_p)\rho_{pq\sigma}.
\]

This is easy to see because \( \rho_{pq\sigma} \) just "moves" a single particle in the Fermi sea to a different momentum, and therefore we only need to account for the energy difference between the initial and final configurations.

The commutator with the interaction is more difficult to handle. The claim of Bohm and Pines is that
\[
[\rho_{pq\sigma}, \frac{1}{2V} \sum_{q'} \rho_{q'q} \rho_{-q}] \approx 2(f_p - f_{p+q})v(q)\rho_{q}. \text{ (RPA approximation)}.
\]

Here, \( f_p \) is the Fermi-Dirac distribution for \textit{noninteracting} fermions at wavevector \( p \). In the noninteracting case, \( c_{p\sigma}^\dagger c_{p\sigma} = Vf_p \). If you look at the above form, it looks like we computed the commutator and then replaced one of the \( \rho \)'s with the noninteracting distribution function, but not the second \( \rho \). That is why the interaction parameter \( v(q) \) survived the RPA method. Let us see how this is justified.

Without any approximations, we have
\[
[\rho_{pq\sigma}, \frac{1}{2V} \sum_{q'} \rho_{q'q} \rho_{-q'}] = \frac{1}{2V} \sum_{q'\sigma} v(q') \rho_{p,q+q',\sigma} - \rho_{p-q',q',\sigma} \rho_{q-q',\sigma}.
\]

Bohm and Pines argued that we can approximate \( \sum_{q'\sigma} \rho_{p,q+q',\sigma} \approx \sum_\sigma \rho_{p,0,\sigma} \) and \( \sum_{q'\sigma} \rho_{p-q',q+q',\sigma} \approx \sum_\sigma \rho_{p-q',0,\sigma} \). In other words, only the density operators which do not add or subtract momentum from the system contribute, and we ignore "off-diagonal" density operators.
This follows by assuming that the low-energy states of an electron gas contain highly delocalized electrons, so the wavefunction \( \Psi(x_1, \ldots, x_N) \) in

\[
|\Psi⟩ = \frac{1}{N!} \int_{x_1, \ldots, x_N} \Psi(x_1, \ldots, x_N) |x_1, \ldots, x_N⟩
\]
is a smooth and slowly-varying function of the spatial arguments. If this is true, then the Fourier-transform factor \( e^{-i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}_i)} \) varies much faster than the function itself for large \( |\mathbf{r} - \mathbf{r}_i| \). So, for example,

\[
\rho_{pq}\sigma |\Psi⟩ = \frac{1}{N!} \int_{x_1, \ldots, x_N, \mathbf{r}} \sum_{i=1}^{N} e^{i\mathbf{p} \cdot (\mathbf{r} - \mathbf{x}_i)} e^{-i\mathbf{q} \cdot \mathbf{x}_i} \Psi(x_1, \ldots, x_{i-1}, \mathbf{r}, x_{i+1}, \ldots, x_N) |x_1 \cdot x_{i-1} \mathbf{r} x_{i+1} \cdots x_N⟩
\]

\[
= \frac{\Delta V}{N!} \int_{x_1, \ldots, x_N} \sum_{i=1}^{N} e^{-i\mathbf{q} \cdot \mathbf{x}_i} \Psi(x_1, \ldots, x_N) |x_1 \cdots x_{i-1} x_i x_{i+1} \cdots x_N⟩.
\]

In the second equality, we integrated over \( \mathbf{r} \) with stationary phase approximation. The “standard deviation” of the Gaussian distribution produced a factor of \( \Delta V \) in the front, which is a volume element which scales for \( e^{i\mathbf{p} \cdot \mathbf{x}} \) like \( \frac{1}{p_p p_q p_z} \) or something, and we also replaced \( \mathbf{r} \rightarrow \mathbf{x}_i \) in the Fock state because that is where the distribution is peaked. If you look at the second integral, the integral over \( \mathbf{x}_i \) approximately enforces a delta-function \( \delta(\mathbf{q}) \), since \( |x_1 \cdots x_{i-1} x_i x_{i+1} \cdots x_N⟩ \) is slowly-varying in \( \mathbf{x}_i \). We conclude that

\[
\rho_{pq}\sigma |\Psi⟩ \text{ is very small unless } \mathbf{q} = 0.
\]

Therefore, we can rewrite the commutator in question as

\[
\frac{1}{2V} \sum_{\mathbf{q}'\sigma} v(\mathbf{q'}) [\rho_{\mathbf{p}, \mathbf{q}+\mathbf{q}', \sigma} - \rho_{\mathbf{p}-\mathbf{q}, \mathbf{q}'+\mathbf{q}, \sigma}] \approx \frac{1}{2V} \sum_{\sigma} v(\mathbf{q}) [c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma} - c^\dagger_{\mathbf{p}+\mathbf{q}, \sigma} c_{\mathbf{p}+\mathbf{q}, \sigma} ; \rho_{\mathbf{q}}].
\]

We now make the approximation that \( c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma} \approx V f_p \). This is actually something I can believe, and it follows because we expect even very strong interactions to not distort the shape of the Fermi sea by too much. For example, this is the central assumption behind Landau Fermi-liquid theory. The result claimed earlier follows immediately. (We can replace \( c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma} \) with a macroscopic parameter because it carries zero momentum. We cannot replace \( \rho_{\mathbf{q}} \) with a macroscopic parameter because it carries momentum and hence is off-diagonal in \( \mathbf{k} \)-space.)

I think a good question to ask is why the many-body state \( |\Psi⟩ \) is relevant here. When we were evolving the density, did we ever have to invoke \( |\Psi⟩ \)?

Answer: Technically, no. Realistically, yes, because we actually take the many-body expectation \( \langle O \rangle \) of everything at the end. So I guess this makes sense.

### 3 The finish line

The approximate equation of motion for the response function is

\[
\frac{i}{\hbar} \frac{d\chi_{\mathbf{p}}(\mathbf{q}, t)}{dt} = 2\delta(t)(f_{\mathbf{p}} - f_{\mathbf{p}+\mathbf{q}}) + (\epsilon_{\mathbf{p}+\mathbf{q}} - \epsilon_{\mathbf{p}}) \chi_{\mathbf{p}}(\mathbf{q}, t) + \frac{2}{V} v(\mathbf{q})(f_{\mathbf{p}} - f_{\mathbf{p}+\mathbf{q}}) \sum_{\mathbf{p}'} \chi_{\mathbf{p}'}(\mathbf{q}, t).
\]
Going to frequency space and using the right \( \delta(t) \) gives us

\[
\chi_p(q, \omega) = \frac{2(f_p - f_{p+q})}{\omega - \epsilon_{p+q} - \epsilon_p + i0} \left( 1 + \frac{v(q)}{V} \sum_{p'} \chi_{p'}(q, \omega) \right).
\]

Thus,

\[
\chi(q, \omega) = \frac{1}{V} \sum_p \chi_p(q, \omega) = \frac{2}{V} \sum_p \frac{f_p - f_{p+q}}{\omega - \epsilon_{p+q} - \epsilon_p + i0} (1 + v(q)\chi(q, \omega)) = \chi_0(q, \omega)(1 + v(q)\chi(q, \omega)).
\]

This gives the famous response for the response function \( \chi \) in terms of the interaction \( v(q) \) and the noninteracting response function \( \chi_0 \),

\[
\chi(q, \omega) = \frac{\chi_0(q, \omega)}{1 - v(q)\chi_0(q, \omega)}, \text{ where } v(q) = \frac{4\pi e^2}{q^2}.
\]

This can be used to describe Thomas-Fermi screening, plasma oscillations, etc.