These are short notes on the projection formalism, Mori-Zwanzig projector formalism and the Linblad quantum kinetic equation. They are different techniques for writing down the nonequilibrium dynamics for a system often coupled to a bath, but are very related. That’s why they are all in the same notes. In fact, sometimes when people say “Zwanzig projector method,” they are really referring to the first formalism I describe. Other times they are referring to the second formalism. I think this is because Zwanzig had a hand in developing both, and they are quite similar anyway.

Before we jump in, I should mention the regular equations of motion method that Eugene Demler likes a lot. The time-evolution of an operator governed by a Hamiltonian $\mathcal{H}$, in the Heisenberg formalism, is

$$i\partial_t \mathcal{O} = [\mathcal{O}, \mathcal{H}].$$

This is if the Hamiltonian is known and is easy to compute for second-quantized Hamiltonians and operators, for example. The Zwanzig projector formalism and Linblad equation are, to a large extent, generalizations of this equation to the case of more complicated Hamiltonians (for example, if the system of interest is in thermal contact with a heat bath) and more complicated densities (for example, if $\rho$ corresponds to a mixed state rather than a pure state).

## Contents

1. **Density operator formalism**
   1.1 Classical dynamics ................................................. 2
   1.2 Quantum formalism ................................................. 2
   1.3 The Liouvillian operator ........................................... 4
   1.4 Projection operator formalism ..................................... 6

2. **Mori-Zwanzig formalism**
   2.1 Notes about Mori-Zwanzig equation .............................. 9
1 Density operator formalism

This follows “Density Operator Formalism” from the Lausanne Lectures by Gert van der Zwan (2016).

1.1 Classical dynamics

Classically, we describe a particle’s behavior with two descriptors: position and momentum. The probability density of find it at some point in phase space is governed by a density

$$\rho(\vec{r},\vec{p},t).$$

The time dependence of this $\rho$ (i.e. how it evolves) follows from the Hamilton equations. It is

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} - \{H,\rho\}$$

where the Poisson bracket is defined

$$\{A,B\} = \sum_i \left( \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial x_i} \frac{\partial A}{\partial p_i} \right).$$

Liouville’s theorem is a conservation law:

$$\frac{d\rho}{dt} = 0 \implies \frac{\partial \rho}{\partial t} = \{H,\rho\}.$$ 

This means that the density in phase space behaves like an incompressible fluid. Weinberg (1995) elucidates that this can be rephrased as a corollary of the fact that the time-translation operator is unitary, and hence probability is conserved. A possible solution is

$$\rho_{eq} = \frac{e^{-\beta H}}{\int_{\vec{p}\vec{x}} e^{-\beta H}},$$

which follows because $\{f(H),H\} = 0$.

1.2 Quantum formalism

Suppose we have a state vector $|\psi\rangle$ which is decomposed into an orthogonal basis as

$$|\psi\rangle = \sum_n \psi_n |n\rangle.$$

Define the density matrix which corresponds to the state $|\psi\rangle$ as

$$\hat{\rho} = |\psi\rangle \langle \psi| \implies \rho_{nm} = \psi_n^* \psi_m \implies \langle A |_{|\psi\rangle} = \text{Tr} (\rho A).$$
The diagonal entries of the density matrix, \( \rho_{nn} \), are interpreted as *populations*. The off-diagonal entries \( \rho_{nm}, n \neq m \), are interpreted as *coherences*. While they do not tell us physical information about where particles are, they *do* tell us about phases and quantum entanglement. Note that the density matrix is a matrix of scalars. The entries of the density matrix are related to the projection of the state vector \( |\psi\rangle \) onto the basis vectors \( |n\rangle \). For example, if we had a quantum state representing a spin-\( \frac{1}{2} \) system,

\[
|\psi\rangle = \cos \theta |0\rangle + \sin \theta e^{i\phi} |1\rangle,
\]

then the density matrix would be

\[
\rho = \begin{pmatrix}
\cos^2 \theta & \cos \theta \sin \theta e^{i\phi} \\
\cos \theta \sin \theta e^{-i\phi} & \sin^2 \theta
\end{pmatrix},
\]

at least at \( t = 0 \). For \( t > 0 \), obviously there is time-evolution.

**Equilibrium density matrix**: What is the equilibrium density matrix? We define it to be when *coherences*, or off-diagonal elements, *vanish*, and diagonal elements become *equilibrium occupations*. For example, for a 2-state system with energies

\[
\mathcal{H}|0\rangle = 0, \mathcal{H}|1\rangle = \epsilon |1\rangle,
\]

the equilibrium density matrix would obviously be

\[
\rho_{\text{eq}} = \frac{1}{1 + e^{\beta \epsilon}} \begin{pmatrix} 1 & 0 \\ 0 & e^{\beta \epsilon} \end{pmatrix}.
\]

However, it is *impossible* to find a pure quantum state which reproduces this density matrix! This is because a pure quantum state is a projection operator satisfying \( \rho^2 = \rho \implies \text{Tr}(\rho^2) = 1 \), but in this case we have

\[
\text{Tr}(\rho^2) < 1.
\]

We conclude that the equilibrium density matrix can only represent mixed states. It is an average over all quantum states – a procedure which returns the classical distribution function.

Important conclusion:

In general, “equilibrium” is not, in itself, a *state*. Rather, “equilibrium” is a property of the Hamiltonian. Equivalently, “equilibrium” is an expectation over all states.

The **quantum Liouville equation** for the density matrix is

\[
\dot{\rho} = i[H, \rho].
\]

This follows if you just apply the time-evolution to the definition of \( \rho \) in terms of state vectors. This is the *opposite sign* to the Heisenberg equation of motion for an operator \( \mathcal{O} \),

\[
-i\dot{\mathcal{O}} = [\mathcal{H}, \mathcal{O}].
\]
1.3 The Liouvillian operator

We can recast the time-evolution of the density in terms of something called the **Liouvillian operator**. In classical statistical mechanics, it is defined

\[ L = \sum_i \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial p_i} \right). \]

As you can see, it encodes time evolution because \( L \mathcal{O} = \{ \mathcal{H}, \mathcal{O} \} \). This will be generalized to the quantum case, with commutators instead of Poisson brackets.

Now for the quantum case. Define **Liouville space** to be the space of operators which act on the space containing the wavefunctions. If this isn’t clear, let \( \mathcal{H}_0 \) be the Hilbert space containing the wavefunctions. The operators which act on \( \mathcal{H}_0 \) belong to another Hilbert space, called \( \mathcal{H}_1 \). \( \mathcal{H}_1 \) is called the Liouville space. The inner-product on \( \mathcal{H}_1 \) is defined to be

\[ \langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = \text{Tr} ( \mathcal{O}_1 \mathcal{O}_2 ). \]

Because operators (such as the density operator) are vectors in Liouville space, we can think of \( \rho \) for a two-state system as

\[ | \rho \rangle = \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{pmatrix} \]

although the explicit representation is definitely overkill.

Now we can write a **Quantum Liouville equation** for the evolution of the density matrix. It is

\[ \partial_t | \rho \rangle = -iL | \rho \rangle, \]

where \( L_{ij,kl} = \mathcal{H}_{ik} \delta_{jl} - \mathcal{H}_{lj} \delta_{ik} \).

The relative minus sign between the two terms in the definition of \( L_{ij,kl} \) is a reflection of the commutator \( [\mathcal{H}, \rho] \). Explicitly in index notation,

\[ \partial_t \rho_{ij} = -iL_{ij,kl} \rho_{kl} \]

with Einstein summation convention. This is really weird! To understand it better, let’s write it out explicitly in terms of \( \mathcal{H} \):

\[ i\partial_t \rho_{ij} = \mathcal{H}_{ik} \rho_{kj} - \rho_{il} \mathcal{H}_{lj}. \]

That makes more sense. So the first term is time evolving the first index in \( \rho \), and the second term is time evolving the second index. Also, the matrix elements of \( \mathcal{H} \) are just what we think they are,

\[ \mathcal{H}_{ij} = \langle i | \mathcal{H} | j \rangle. \]

**More about Liouville quantum kinetic equation**: Here are some more good things to know about the Liouville formulation of density time-evolution.

- Because \( L \) is linear in \( \mathcal{H} \), we can often split the Liouvillian into bare and interaction parts, like

\[ i\partial_t | \rho \rangle = (L_0 + L_{\text{int}}) | \rho \rangle. \]
• In the regular Hilbert space $\mathcal{H}_0$ of the wavefunctions, the action of the Liouvillian can be restated as

$$LO = [\mathcal{H}, O].$$

This is just the regular time-evolution,

$$i\partial_t \rho = L\rho = [\mathcal{H}, \rho].$$

• Unsurprisingly, we formally represent the time-dependent solution for the density matrix $\rho$ as an exponentiation of the Liouvillian.

If the Hamiltonian is time-independent, then this is easy:

$$|\rho(t)\rangle = e^{-iLt}|\rho_0\rangle \text{ if } \mathcal{H} \text{ time-independent.}$$

If the Hamiltonian has a time-dependent part, especially an interaction, then typically we write a Dyson series just like in regular QFT. Suppose the Hamiltonian is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}(t).$$

Although $\mathcal{H}_0$ is time-independent, that does not mean we can commute it past $\mathcal{H}_{\text{int}}$. The formal solution, if $|\rho\rangle$ was initially in state $|\rho_0\rangle$, is given by the Dyson identity or Duhamel identity:

$$|\rho(t)\rangle = e^{-iL_0t}|\rho_0\rangle - ie^{-iL_0t} \int_0^t d\tau e^{iL_0(\tau-t)}L_{\text{int}}(\tau)|\rho(\tau)\rangle.$$

(To check that this is indeed the correct solution, differentiate with respect to $t$.) Note that the $|\rho\rangle$ on the very RHS is time-dependent. This is something we can solve using Born series, but we have to be careful to leave everything in time-ordered form. For example, to second order the Born series (or Dyson series, because this is time-ordered) is

$$|\rho(t)\rangle = U(t, 0)|\rho_0\rangle$$

where

$$U(t, 0) = e^{-iL_0t} - i \int_0^t d\tau e^{iL_0(\tau-t)}L_{\text{int}}(\tau) - \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 e^{iL_0(\tau_1-\tau)}L_{\text{int}}(\tau) e^{iL_0(\tau_2-\tau_1)}L_{\text{int}}(\tau_2) + \cdots.$$  

• Formally, the time-evolution of the expectation of an operator $O$ can be expressed with the help of the Liouvillian,

$$\langle O(t) \rangle = \text{Tr } (O \rho(t)) = \text{Tr } (O e^{-iLt\rho_0}).$$

• Finally, let’s reflect on the strengths of the Liouvillian formalism. Why is it useful when we can just evolve the eigenstates? (1) For pure states, Schrodinger and Liouville forms are equivalent. However, the Hamiltonian is a unitary and so Hamiltonian dynamics will always take a pure state to a pure state. Often, we want to go from a pure state to a mixed state like an equilibrium density matrix. (2) the Liouvillian form helps us go to the impure states which have no Hilbert space equivalent, and hence cannot be handled in the Schrodinger formalism. (3) It’s really easy to trace over the density matrix, which gives us the reduced density matrix. Equivalently, Liouville Hilbert space is much bigger than the regular wavefunction Hilbert space. (4) The approach to equilibrium can be implemented by coupling to a heat bath and then integrating out the heat bath. Generally, there is no approach to equilibrium without introducing a decay mechanism.
1.4 Projection operator formalism

As we saw in the last section, the Liouvillian formulation of quantum evolution could, in principle, account for the approach to an equilibrium density matrix, one which is physically meaningful but is not allowed in the wavefunction Hilbert space because it corresponds to a mixed state. The projection operator formalism is a way to obtain this approach to equilibrium. Essentially, the system is coupled to a bath, and the bath variables are integrated out.

Let there be a system with system operators \( \{s_k\} \) and a bath with operators \( \{b_l\} \). The total Hamiltonian is

\[
\mathcal{H} = \mathcal{H}_s + \mathcal{H}_b + \mathcal{H}_{\text{int}},
\]

or the sum of system, bath, and system-bath interaction energies, respectively. Often, interaction Hamiltonian is assumed to be linear in the system and bath operators,

\[
\mathcal{H}_{\text{int}} = \sum_{kl} \alpha_{kl} s_k b_l,
\]

but this will not be required for most of the manipulations we do. This is an assumption of the famous Caldeira-Leggett model. There is a good review of it by Caldeira himself on Scholarpedia.

Let the total density operator, which describes the state of both the system and bath, be \( \rho(s,b,t) \). Let the reduced density operator which describes just the system be \( \sigma(s,t) \), where

\[
\sigma(s,t) = \text{Tr}_b(\rho(s,b,t)).
\]

If \( \bar{\sigma}_{eq}(b) \) is the bath equilibrium density operator,

\[
\bar{\sigma}_{eq}(b) = \text{Tr}_s(\rho_{eq}(s,b)),
\]

then we define the projection operator \( \mathcal{P} \) to have action

\[
\mathcal{P}\rho(s,b,t) = \bar{\sigma}_{eq}(b)\sigma(s,t).
\]

Why this projection operator? This makes sense if we think that the bath is always in equilibrium, but the system is not. This is an assumption made, for example, in the Caldeira-Leggett model and extensively used in Kamenev’s book. To have convenient notation, we also define

\[
(1 - \mathcal{P})\rho(s,b,t) = \eta(s,b,t).
\]

We often assume that before the experiment, the system \( s \) and bath \( b \) are in equilibrium but \( L_{\text{int}}(t) = 0 \) for \( t < 0 \), so there is no system-bath coupling. In that case,

\[
\eta(s,b,0) = \rho_{eq}(s,b) - \sigma_{eq}(s)\sigma_{eq}(b).
\]

Okay, now we are ready to do some nonequilibrium time evolution! We start with

\[
\partial_t \rho(b,s,t) = -iL\rho(b,s,t).
\]

Separate this equation into two parts by using \( \partial_t \rho = -iL\mathcal{P}\rho - iL(1 - \mathcal{P})\rho \),

\[
\partial_t (\mathcal{P}\rho(b,s,t)) = -i\mathcal{P}L\rho(b,s,t) - \mathcal{P}L(1 - \mathcal{P})\rho(b,s,t)
\]
\[ \partial_t((1 - \mathcal{P})\rho(b, s, t)) = -i(1 - \mathcal{P})L\mathcal{P}\rho(b, s, t) - i(1 - \mathcal{P})L(1 - \mathcal{P})\rho(b, s, t). \]

Because the bath is always in equilibrium, we can divide many of the \( \tilde{\sigma} \) factors out. The result is

\[ \partial_t\sigma(s, t) = -iL_s\sigma(s, t) - i\text{Tr}_b(\mathcal{L}_{\text{int}}\sigma(s, b, t)) \]

\[ \partial_t\eta(s, b, t) = -i\mathcal{L}_{\text{int}}\tilde{\sigma}\text{eq}(b)\sigma(s, t) - i(L_s + L_b + (1 - \mathcal{P})\mathcal{L}_{\text{int}})\eta(s, b, t). \]

To get rid of the \( \eta(s, b, t) \) in the first equation (which is the important one), we solve for \( \eta(s, b, t) \) from the second equation. Formally, this is

\[ \eta(s, b, t) = e^{-i(L_s + L_b + (1 - \mathcal{P})\mathcal{L}_{\text{int}})t}\eta(s, b, 0) - i\int_0^td\tau e^{-i(L_s + L_b + (1 - \mathcal{P})\mathcal{L}_{\text{int}})(t - \tau)}\mathcal{L}_{\text{int}}\tilde{\sigma}\text{eq}(b)\sigma(s, \tau) \]

which can be substituted into the equation for \( \partial_t\sigma(s, t) \). So far, everything is exact. There are no “slow” or “fast” variables here, unlike in the Mori-Zwanzig method. The initial condition is a source term (kind of like a transient) which should vanish for long times and usually does not contribute to the approach to equilibrium. If we neglect the initial condition term (the term with \( \eta(s, b, 0) \)), then to second order the equation becomes

\[ \partial_t\sigma(s, t) = -iL_s\sigma(s, t) - \int_0^td\tau\text{Tr}_b(\mathcal{L}_{\text{int}}e^{-2\pi i(L_s + L_b)(t - \tau)}\mathcal{L}_{\text{int}}\tilde{\rho}_{\text{eq}}(b)\sigma(s, \tau)) \]

or equivalently, if we want to think of \( \rho, \sigma \) as operators instead of vectors, this becomes

\[ \partial_t\sigma(s, t) = -iL_s\sigma(s, t) - \int_0^td\tau\text{Tr}_b[\mathcal{H}_{\text{int}}, e^{-2\pi i(L_s + L_b)(t - \tau)}][\mathcal{H}_{\text{int}}, \tilde{\rho}_{\text{eq}}(b)\sigma(s, \tau)]. \]

In the case that \( \mathcal{H}_{\text{int}} = \sum_{kl}\alpha_{kl}s_kb_l \), the complicated memory term becomes

\[ \sum_{kl}\sum_{k'l'}\alpha_{kl}\alpha_{k'l'}[\langle s_k e^{-iL_s(t - \tau)}s_{k'}\sigma(\tau)\rangle\langle [b_l, b_{l'}(t - \tau)]\rangle_{\text{eq}} + [s_k, e^{-iL_s(t - \tau)}s_{k'}\sigma(\tau)]\langle b_l b_{l'}(t - \tau)\rangle_{\text{eq}}], \]

where we used the abbreviations

\[ \langle [b_l, b_{l'}(t - \tau)]\rangle_{\text{eq}} = \text{Tr}_b[b_l e^{-iL_b(t - \tau)}[b_{l'}, \tilde{\rho}_{\text{eq}}(b)]] \text{ and } \langle b_l b_{l'}(t - \tau)\rangle_{\text{eq}} = \text{Tr}_b[b_l e^{-iL_b(t - \tau)}b_{l'}\tilde{\rho}_{\text{eq}}(b)]. \]

Above, we replaced \( \tilde{\sigma} \rightarrow \tilde{\rho} \) for the bath. This is quite a useful form. For an example application to Hydrogen quantum diffusion on metal surfaces, see Pouthier and Light, J. Chem. Phys. 113 3 pg. 1204-1215 (2000). Obviously, the difficulty is in calculating the nasty second part of the above equation.

Interestingly, in the paper by Pouthier and Light, they really assumed nothing about the characteristics of the bath, and all their final expressions were still in traces over the bath variables. Conclusion: the projector technique can accommodate all kinds of heat baths.

Let’s return to the above equation and think about each term more. The first part of the equation,

\[ \partial_t\sigma(s, t) = -iL_s\sigma(s, t), \]

is just the evolution of the density of the system variables by the system Hamiltonian, \( \mathcal{H}_s \). This is the same as the equation of motion method. The second term is a memory term, so named because the integral

\[ \int_0^td\tau\text{Tr}_b(\mathcal{L}_{\text{int}}e^{-2\pi i(L_s + L_b)(t - \tau)}\mathcal{L}_{\text{int}}\tilde{\rho}_{\text{eq}}(b)\sigma(s, \tau)) \]

depends on all times \( \tau \) for \( 0 \leq \tau \leq t \). The \( L_s + L_b \) term is there to evolve both the system and the bath. We split the \( \mathcal{L}_{\text{int}} \) terms because they would be difficult to handle inside an exponential.
2 Mori-Zwanzig formalism

The Mori-Zwanzig formalism is a method used in non-equilibrium statistical mechanics and is a technique of deriving a generalized Langevin equation (GLE) which describes the non-equilibrium evolution of any functions defined on the phase-space of the microscopic system. Typically these are coarse-grained variables.

We will follow https://en.wikipedia.org/wiki/Zwanzig_projection_operator and also the seminal paper by Zwanzig, Phys. Rev. 124 4 pg 983-992 (1961) and also the exposition www.math.nyu.edu/faculty/kleeman/Slecture11.pdf. This exposition describes how to split the “slow” degrees of freedom from the “fast” degrees of freedom. The idea is that the set of functions in phase space is a linear space \( H \), and the set of slow functions in phase space is a subspace \( H_{\text{slow}} \). The idea of the Zwanzig projector is to define a projection function \( P \) such that

\[
P(H) = H_{\text{slow}},
\]

so \( P \) projects away the fast degrees of freedom.

How does this work? Consider the regular equation of motion for the slow variable \( A \) (for simplicity, we will assume it is a scalar)

\[
\partial_t A = i L A \quad \Rightarrow \quad A(t) = e^{iLt} A(0).
\]

Let’s define our projector later. I just assume I have a projector \( P \), based on an inner product \( \langle f, g \rangle \); the projection is then

\[
P X = \frac{\langle A, X \rangle}{\langle A, A \rangle} A.
\]

More on the projection in the next section. The operator identity (see the previous sections)

\[
e^{iLt} = e^{i(1-P)Lt} + \int_0^t ds e^{iL(t-s)} P L e^{i(1-P)Ls}
\]

applied to \((1 - P)iLA\) gives the generalized Langevin equation

\[
\dot{A}(t) = \Omega A(t) + \int_0^t ds K(s) A(t-s) + F(t)
\]

where \( K \) is the memory function, \( F \) is the random force, and \( \Omega \) is the frequency matrix. It’s important to remember that the random force is not actually a random force – the time-evolution is in fact deterministic, but because we have projected the fast components away, we have to account for their effects with \( F(t) \) which functions kind of like a random force. The expressions for these things are

\[
\Omega = \frac{\langle iLA, A \rangle}{\langle A, A \rangle}, F(t) = e^{i(1-P)Lt}(1 - P)iLA, K(t) = \frac{\langle iLF(t), A \rangle}{\langle A, A \rangle}.
\]

2.0.1 Derivation

Define \( Q = 1 - P \). The equation of motion is

\[
\partial_t A = e^{iLt} LA = ie^{iLt}(Q + P) LA.
\]
It’s easy to show that $e^{iLt}PLA = \Omega A(t)$. We now have the equation
\[ \partial_t A = \Omega A(t) + ie^{iLt}QLA. \]
The Dyson identity gives us
\[ ie^{iLt}QLA = e^{iQLt}QiLA + i \int_0^t ds e^{iL(t-s)}PL e^{QLs}QLA. \]

Then, we define
\[ F(t) = ie^{iQLt}QLA \text{ and } iPL e^{iQLt}QL = -K(t). \]
This gives us Mori-Zwanzig equation. We also used the fact that there is an evolution backwards in time,
\[ e^{iL(t-t_1)}K(t_1)A = K(t_1)A(t-t_1). \]

### 2.1 Notes about Mori-Zwanzig equation
- The stochastic force and the slow operator $A$ are orthogonal:
  \[ \langle F(t), A \rangle = \langle ie^{iQLt}QLA, A \rangle = \langle QF(t), A \rangle = 0. \]
  Of course the stochastic force and the slow operator are orthogonal, since $\mathcal{H}_{\text{slow}} \perp \mathcal{H}_{\text{fast}}$; $A$ lives in the former and $F$ lives in the latter.
- The memory integral $\int_0^t ds K(s)A(t-s)$ measures correlations between the observable $A(t)$ and the noise $F(t)$. (Note that the noise $F(t)$ is inside the definition of $K(s)$. ) In the business, this is known as the convolution term.
  The equation as written is, in fact, exact. The difficulty in solving it is the convolution term; we can make the Markovian approximation by assuming that $A$ does not evolve very much on the timescale of $K$, so maybe it’s not so bad to approximate $A(t-s) \approx A(t)$ and therefore
  \[ \dot{A}(t) \approx \Omega A(t) + A(t) \int_0^\infty ds K(s) + F(t). \]

### 2.2 Zwanzig’s projector, Mori’s projector, and the inner product
Apparently, Mori’s inner product is
\[ \langle X, Y \rangle = \beta^{-1} \int_0^\beta d\alpha \text{Tr}[\rho X e^{-\alpha \mathcal{H}} Ye^{\alpha \mathcal{H}}]. \]
Here, $\beta$ is the inverse temperature. It looks like a time evolution but in imaginary time. Maybe there is some connection to the Matsubara sum, etc.
Zwanzig’s inner product is defined in phase space $\Gamma$ and is
\[ \langle X, Y \rangle = \int d\Gamma \rho_0(\Gamma)X(\Gamma)Y(\Gamma). \]
I think that the integral gets a cutoff at high energies. This can be when the momentum $p \in \Gamma$ is very large, for example.
So, it looks like the inner product is just like a regular inner product with a thermal weighting. I think there’s some freedom in exactly how you define it.
2.2.1 Frequency projection operator

I have an alternative way of thinking about things which is a lot simpler. Consider a very bumpy function \( f(t) \) on a time interval \([0, T]\). We want to project it onto a smoother function which evolves more slowly.

The way to think of this is to introduce a frequency cutoff. So maybe my original Hilbert space was

\[
\mathcal{H}_0 : f(t) = \sum_{n=0}^{\infty} a_n e^{i\frac{2\pi n}{T} t} \in \mathcal{H}_0
\]

but my smaller Hilbert space is

\[
\mathcal{H}_1 : f(t) = \sum_{n=0}^{N_1} a_n e^{i\frac{2\pi n}{T} t} \in \mathcal{H}_1
\]

where I introduced a frequency cutoff. The choice of frequency cutoff governs how smooth my functions are.

On this \( L_2 \) space, the inner product is obvious. Plus, the projection operator can be written as

\[
P f(t) = \frac{1}{T} \sum_{n=0}^{N_1} \int_0^T e^{-i\frac{2\pi n}{T} t} f(t).
\]

Obviously, the rest of the space has its own projection operator,

\[
1 - P = Q = \frac{1}{T} \sum_{n=N_1+1}^{\infty} \int_0^T e^{-i\frac{2\pi n}{T} t}.
\]

The higher I raise the cutoff \( N_1 \), the more the noise term \( F(t) \) resembles white noise. Heuristically speaking, the lowest possible frequency of \( F(t) \) is like \( N_1/T \). So, the correlation time of the noise \( F \) is roughly \( T/N_1 \).